1. If F and f are continuous functions such that $F'(x) = f(x)$ for all x, then $\int_a^b f(x)\,dx$ is

(A) $F'(a) - F'(b)$

(B) $F'(b) - F'(a)$

(C) $F(a) - F(b)$

(D) $F(b) - F(a)$

(E) None of the above

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,dx$$

$$= F(x)\Big|_a^b$$

$$= F(b) - F(a)$$

2. If $F(x) = \int_0^x \sqrt{t^3+1}\,dt$, then $F'(2) =$

(A) -3 (B) -2 (C) 2 (D) 3 (E) 18

Fundamental Theorem of Calculus

$$F'(x) = \sqrt{x^3+1}$$

$$F'(2) = \sqrt{2^3+1}$$

$$= \sqrt{9} = 3$$

3. The expression $\frac{1}{20}\left(\sqrt{\frac{1}{20}} + \sqrt{\frac{2}{20}} + \sqrt{\frac{3}{20}} + \cdots + \sqrt{\frac{20}{20}}\right)$ is a Riemann sum approximation for

(A) $\int_0^1 \sqrt{\frac{x}{20}}\,dx$

(B) $\int_0^1 \sqrt{x}\,dx$

(C) $\frac{1}{20}\int_0^1 \sqrt{\frac{x}{20}}\,dx$

(D) $\frac{1}{20}\int_0^1 \sqrt{x}\,dx$

(E) $\frac{1}{20}\int_0^{20} \sqrt{x}\,dx$

w

f(x) must be $\sqrt{x}$

Riemann Sum Equation

$$w\left(f(0) + f(0+w) + f(0+2w) + \cdots\right)$$

$$w = \frac{b-a}{n}$$

~~Normal~~ Integral Form

$$\int_0^1 \sqrt{x}\,dx$$

$$w = \frac{1-0}{20}$$

$b=1,\ a=0,\ n=20$

x	0	1	2	3	4
$f(x)$	3	3	5	6	13

$W = \frac{b-a}{n} = \frac{4-0}{4}$

4. A table of values for a continuous function f is shown above. If four equal subintervals of $[0,4]$ are used, which of the following is the trapezoidal approximation of $\int_0^4 f(x)\,dx$?

(A) 14 (B) 22 (C) 28 (D) 40 (E) 44

$$\int_0^4 f(x)\,dx \approx \frac{1}{2} w\left(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right)$$

$$= \frac{1}{2}(1)\left(3 + 2(3) + 2(5) + 2(6) + 13\right)$$

$$= 22$$

x	2	4	6	8
$f(x)$	10	30	40	50

5. The function f is continuous on the closed interval $[2,8]$ and has values that are given in the table above. Using three equal subintervals, what is the left Riemann sum approximation of $\int_2^8 f(x)\,dx$?

(A) 110 (B) 130 (C) 160 (D) 180 (E) 200

$$\int_2^8 f(x)\,dx \approx w\left(f(2) + f(4) + f(6)\right)$$

$$= 2(10 + 30 + 40)$$

$$= 160$$

$W = \frac{b-a}{n}$

$W = \frac{8-2}{3}$

$W = 2$

6. $\int_0^3 (4x-3)\ dx$

(A) 0 (B) 9 (C) 15 (D) 20 (E) 29

$= \left(4\left(\frac{x^2}{2}\right)-3x\right)\Big|_0^3$

$= \left(2(3)^2-3(3)\right)-\left(2(0)^2-3(0)\right)$

$= (18-9)-0$

$= 9$

7. $\int_0^8 \frac{dx}{\sqrt{1+x}} =$

(A) $4\sqrt{2}$ (B) $\frac{3}{2}$ (C) 2 (D) 4 (E) 6

$= \int_0^8 (1+x)^{-1/2}\, dx$

$= \int_0^8 u^{-1/2}\, du$

$= \frac{u^{-1/2+1}}{-1/2+1}\Big|_0^8$

$= 2u^{1/2}\Big|_0^8$

$= 2\sqrt{1+x}\Big|_0^8$

$= (2\sqrt{9})-(2\sqrt{1})$

$= 6-2$

$= 4$

$u = 1+x$

$\frac{du}{dx} = 1$

$dx = du$

8. $\int \frac{x^2}{e^{x^3}}\,dx =$

(A) $-\frac{1}{3}\ln e^{x^3} + C$ (B) $-\frac{e^{x^3}}{3} + C$ (C) $-\frac{1}{3e^{x^3}} + C$ (D) $\frac{1}{3}\ln e^{x^3} + C$ (E) $\frac{x^3}{3e^{x^3}} + C$

$= \int x^2 e^{-x^3}\,dx$ $\quad u = -x^3$

$= \int \cancel{x^2} e^u \left(\frac{du}{-3\cancel{x^2}}\right)$ $\quad \frac{du}{dx} = -3x^2$

$= -\frac{1}{3}e^u + C$ $\quad dx = \frac{du}{-3x^2}$

$= -\frac{1}{3}e^{-x^3} + C$ $= -\frac{1}{3e^{x^3}} + C$

9. $\int_0^{\frac{\pi}{2}} 2\sec^2\left(\frac{x}{2}\right)\,dx =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 4

$= \int_0^{\frac{\pi}{2}} 2\sec^2(u)(2\,du)$ $\quad u = \frac{x}{2}$

$= 4\int_0^{\frac{\pi}{2}} \sec^2 u\,du$ $\quad \frac{du}{dx} = \frac{1}{2}$

$= 4\tan(u)\Big|_0^{\frac{\pi}{2}}$ $\quad dx = 2\,du$

$= 4\tan\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}}$

$= 4\left(\tan\left(\frac{\pi}{4}\right) - \tan(0)\right)$ $= 4(1-0)$

$= 4$

10. $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{2\cos(2x)}{\sin(2x)}\,dx =$

(A) $-\ln\frac{\sqrt{2}}{2}$ (B) $\ln\frac{1}{2}$ (C) $\ln\sqrt{3}$ (D) $\ln\frac{\sqrt{3}}{2}$ (E) $\ln e$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\cancel{2\cos(2x)}}{u}\left(\frac{du}{\cancel{2\cos(2x)}}\right)$$

$u = \sin(2x)$

$\frac{du}{dx} = \cos(2x)\cdot 2$

$dx = \frac{du}{2\cos(2x)}$

$$= \ln|u|\Big|_{\pi/8}^{\pi/4}$$

$$= \ln|\sin(2x)|\Big|_{\pi/8}^{\pi/4} = \ln\left|\sin\left(\frac{\pi}{2}\right)\right| - \ln\left|\sin\left(\frac{\pi}{4}\right)\right|$$

$$= \ln|1| - \ln\left|\frac{\sqrt{2}}{2}\right| = 0 - \ln\left(\frac{\sqrt{2}}{2}\right)$$

11. If f is a linear function and $0 < a < b$, then $\int_a^b f''(x)\,dx =$

(A) 0 (B) 1 (C) $\frac{ab}{2}$ (D) $b-a$ (E) $\frac{b^2-a^2}{2}$

If $f(x)$ is linear, $f''(x) = 0$

(no concavity)

$$\int_a^b 0\,dx$$

$$= 0\Big|_a^b$$

$$= 0$$

12. $\int_1^e \left(\frac{x^2-1}{x}\right) dx =$

(A) $e - \frac{1}{e}$ (B) $e^2 - e$ (C) $\frac{e^2}{2} - e + \frac{1}{2}$ (D) $e^2 - 2$ (E) $\frac{e^2}{2} - \frac{3}{2}$

$= \int_1^e \left(x - \frac{1}{x}\right) dx$

$= \left(\frac{x^2}{2} - \ln|x|\right)\Big|_1^e$

$= \left(\frac{e^2}{2} - \ln(e)\right) - \left(\frac{1}{2} - \ln(1)\right)$

$= \frac{e^2}{2} - 1 - \frac{1}{2}$

$= \frac{e^2}{2} - \frac{3}{2}$

13. What are all values of k for which $\int_{-3}^{k} x^2 dx = 0$?

(A) -3 (B) 0 (C) 3 (D) -3 and 3 (E) $-3, 0,$ and 3

$= \frac{x^3}{3}\Big|_{-3}^{k}$

$= \frac{k^3}{3} - \left(\frac{-3^3}{3}\right)$

$0 = \frac{k^3}{3} + 9$

$-9 = \frac{k^3}{3}$

$\sqrt[3]{-27} = \sqrt[3]{k^3}$

$k = -3$

Use the following information for questions 14-18. Let $\int_0^3 f(x) = 2$, $\int_0^6 f(x) = 8$, and $\int_3^6 g(x) = 3$

14. $\int_3^6 f(x)\,dx =$

(A) 2 (B) 8 (C) -8 (D) 6 (E) Cannot be determined

$= \int_0^6 f(x)\,dx - \int_0^3 f(x)\,dx = 8 - 2 = 6$

15. $\int_6^3 3g(x)\,dx =$

(A) -3 (B) 9 (C) -9 (D) 3 (E) Cannot be determined

$= -3\int_3^6 g(x)\,dx = -3(3) = -9$

16. $\int_0^3 2(f(x) + 1)\,dx =$

(A) 6 (B) 5 (C) 4 (D) 10 (E) Cannot be determined

$= 2\int_0^3 f(x)\,dx + 2\int_0^3 1\,dx$

$= 2(2) + 2x\Big|_0^3 \rightarrow = 4 + 2(3-0)$

$= 10$

17. $\int_3^6 (f(x) * g(x))\,dx =$

(A) 3 (B) 8 (C) -1 (D) 5 (E) Cannot be determined

Not a property of integrals

18. $\int_1^1 3g(x)\,dx =$

(A) 0 (B) 3 (C) 9 (D) -3 (E) Cannot be determined

same bounds $= 0$

19. Given $f(x) = sec^2 x + \frac{1}{\pi}$ and $F(\pi) = 4$, where $F(x) = \int f(x)\,dx$. What is $\int f(x)\,dx$?

(A) $-\cos x + \frac{x}{\pi} + 2$ (B) $\cos x + \frac{x}{\pi} + 4$ (C) $\tan x + \frac{x}{\pi} + 3$

(D) $\sin x + \frac{x}{\pi} + 3$ (E) $-\sin x + \frac{x}{\pi} + 3$

$$\int \left(\sec^2 x + \frac{1}{\pi}\right) dx$$

$$= \tan x + \frac{1}{\pi} x + C$$

$$F(\pi) = 4$$

$$4 = \tan(\pi) + \frac{1}{\cancel{\pi}}(\cancel{\pi}) + C$$

$$4 = 0 + 1 + C \Rightarrow C = 3$$

$$\tan x + \frac{1}{\pi} x + 3$$

20. What is the value of k for which $\int_0^k \frac{3}{1+x^2}\,dx = \frac{3\pi}{4}$?

(A) 0 (B) 1 (C) 2 (D) -3 (E) 4

$$\int_0^k \frac{3}{1+x^2}\,dx$$

$$= 3\tan^{-1}(x)\Big|_0^k$$

$$= 3\left(\tan^{-1}(k) - \tan^{-1}(0)\right)$$

$$\frac{3\pi}{4} = 3\tan^{-1}(k)$$

$$\frac{\pi}{4} = \tan^{-1}(k)$$

$$\tan\left(\frac{\pi}{4}\right) = k$$

$$k = 1$$

21. $\int \frac{-4e^{4/x}}{x^2}\,dx =$

(A) $\frac{-4e^{4/x}}{x^2} + C$

(B) $-4e^{4/x} + C$

(C) $e^{4/x} + C$

(D) $\frac{e^{4/x}}{x} + C$

(E) $4e^4 - e^x + C$

$$\int \frac{-4e^{4x^{-1}}}{x^2}\,dx$$

$$u = 4x^{-1}$$

$$\frac{du}{dx} = -4x^{-2}$$

$$du = \frac{-4}{x^2}\,dx$$

$$dx = \frac{x^2}{-4}(du)$$

$$= \int \frac{\cancel{-4}e^u}{\cancel{x^2}}\left(\frac{\cancel{x^2}}{\cancel{-4}}\,du\right)$$

$$= \int e^u\,du$$

$$= e^u + C$$

$$= e^{4/x} + C$$

Free Response

Time $t\ (seconds)$	0	10	20	30	40	50	60	70	80
Velocity $v(t)\ (m/s)$	24	32	44	82	118	164	222	286	348

1. A bullet train is accelerating and travelling in the positive direction. The velocity $v(t)$ is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above. Use this data to answer the following questions about the bullet train.

   a. Approximate the total distance travelled over the first minute, $\int_0^{60} v(t)\,dt$, using a left Riemann sum with **3 subintervals of equal length**. Include proper units. Is this an overestimate or underestimate?

$$\int_0^{60} v(t)\,dt \approx \frac{60-0}{3}\left(v(0)+v(20)+v(40)\right)$$

$$= 20(24+44+118)$$

$$= 3{,}720 \text{ m}$$

$v(t)$ is increasing

$\Rightarrow$ left Riemann sum is an underestimate

   b. Approximate the total distance travelled over the first minute, $\int_0^{60} v(t)\,dt$, using a right Riemann sum with **3 subintervals of equal length**. Include proper units. Is this an overestimate or underestimate?

$$\int_0^{60} v(t)\,dt \approx \frac{60-0}{3}\left(v(20)+v(40)+v(60)\right)$$

$$= 20(44+118+222)$$

$$= 7{,}680 \text{ m}$$

$v(t)$ is increasing

$\Rightarrow$ right Riemann sum is an overestimate

c. Approximate the total distance travelled over the first minute, $\int_0^{60} v(t)\,dt$, using a midpoint Riemann sum with **3 subintervals of equal length**. Include proper units.

$$\int_0^{60} v(t)\,dt \approx \frac{60-0}{3}\left(v(10)+v(30)+v(50)\right)$$

$$=20\left(32+82+164\right)$$

$$=5{,}560\text{ m}$$

d. Approximate the total distance travelled over the **first 80 seconds**, $\int_0^{80} v(t)\,dt$, using the trapezoidal method with **4 subintervals of equal length**. Include proper units.

$$\int_0^{80} v(t)\,dt \approx \frac{80-0}{4}\left(v(0)+2(v(20))+2(v(40))+ + 2(v(60))+v(80)\right)$$

$$=20\left(24+2(44)+2(118)+2(222)+348\right)$$

$$=22{,}800\text{ m}$$

2. The functions f and g are given by $f(x) = \int_2^x (4t - t^2)\,dt$ and $g(x) = f(e^x)$.
   a. Find $f'(x)$ and $g'(x)$.

Fundamental Theorem of Calculus

$f'(x) = 4x - x^2$

$g(x) = \int_2^{e^x} (4t - t^2)\,dt$

$g'(x) = (4e^x - (e^x)^2) \cdot e^x$

   b. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$

we need a slope and a point

$f'(x)$

$f'(x) = 4x - x^2$

$f'(2) = 4(2) - 2^2$

$= 8 - 4$

$= 4 \leftarrow$ slope

$f(2) = \int_2^2 (4t - t^2)\,dt$

same bounds

$f(2) = 0$

point is $(2, 0)$

$y - y_1 = m(x - x_1)$

$y - 0 = 4(x - 2)$

$y = 4x - 8$

c. Function h is defined by $h(x) = \int_0^{x^2} \left(\frac{2}{\sqrt{t}} - 1\right) dt$. What is the x-value of the relative maximum of $h(x)$?

Find critical points

$$h'(x) = \left(\frac{2}{\sqrt{x^2}} - 1\right)(2x)$$

$$= \left(\frac{2}{x} - 1\right)(2x)$$

$$= \left(\frac{2}{x} - 1\left(\frac{x}{x}\right)\right)(2x)$$

$$= \frac{2-x}{x}(2x)$$

$$= \frac{(2-x)(2x)}{x} \quad \begin{matrix} = 0 \\ = 0 \end{matrix}$$

$x = 0$

$(2-x)(2x) = 0$

$x = 2, 0$

critical points: $x = 2, 0$

$+ \quad + \quad -$

$0 \quad 2 \quad h'(x)$

$h'(-1) = \left(\frac{2}{-1} - 1\right)(2(-1))$

$= (-2-1)(-2)$

$= 6$ (positive)

$h'(1) = \left(\frac{2}{1} - 1\right)(2 \cdot 1)$

$= 1(2)$ (positive)

$h'(3) = \left(\frac{2}{+3} - 1\right)(2(+3))$

$= \cancel{\frac{4}{3}} - \frac{1}{3}(6)$

$= -2$ (negative)

Max at $x = 2$