Dan the Tutor
Learn by Doing

Lesson 1: Introduction to Calculus and Graphs of Limits

What is Calculus?

According to Wikipedia:

> “Calculus, originally called infinitesimal calculus or "the calculus of infinitesimals", is the mathematical study of continuous change, in the same way that geometry is the study of shape and algebra is the study of generalizations of arithmetic operations.”

What's a Limit?

A limit is the value a function/sequence ***approaches*** as the input or index as the input/index moves toward some value.

Limit Notation

$$\lim_{x \to 0} f(x)$$

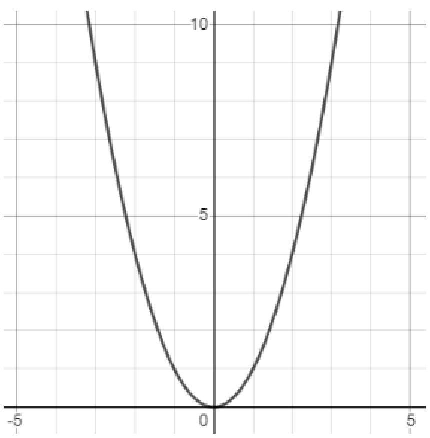
Denotes a limit → $\lim$

The function for which the limit is being evaluated → $f(x)$

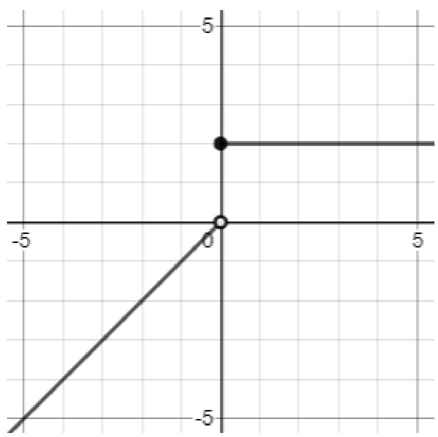
Some index, usually the independent variable → x

The value the index is approaching → 0

"The limit as x approaches 0 of f(x)"



$\lim_{x \to 0} f(x)$



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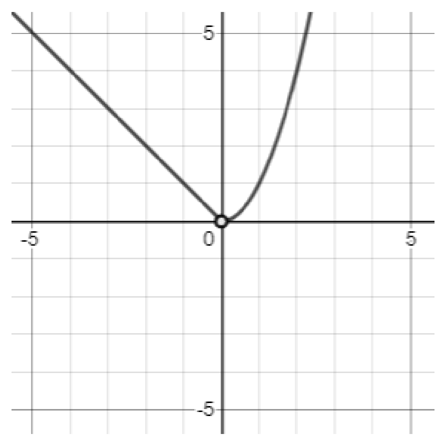


$\lim_{x \to 0} f(x)$

One-sided Limits:

For this situation we will use **right-hand** and **left-hand limits** to help us.

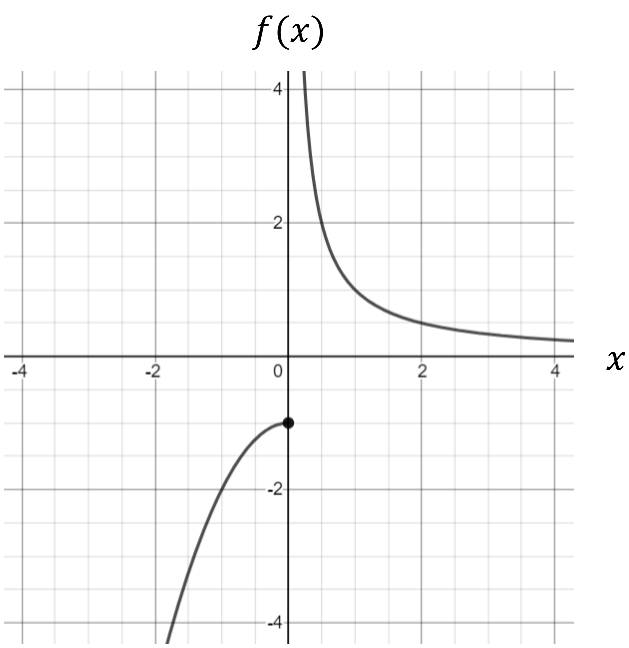
Right-Hand Limit (Definition): The limit of a function at a given point as we approach that point from the right side. Denoted by a "+" superscript.

$$\lim_{x \to a^+} f(x)$$

Left-Hand Limit (Definition): The limit of a function at a given point as we approach that point from the left side. Denoted by a "–" superscript.

$$\lim_{x \to a^-} f(x)$$

If the right- and left-hand limits are not equal to one another at a given point, the limit **does not exist.**

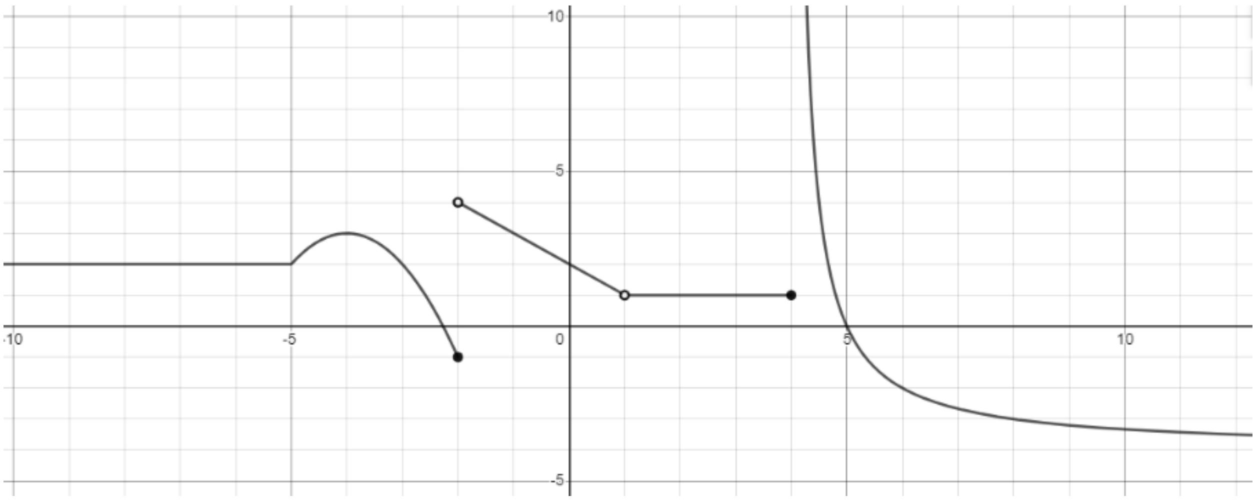


$$\lim_{x \to 0^+} f(x)$$

$$\lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0} f(x)$$

Example Problems:



1. Use the provided graph of $f(x)$ to determine the following limits:

a) $\lim_{x\to -5^-} f(x)$

b) $\lim_{x\to -5^+} f(x)$

c) $\lim_{x\to -5} f(x)$

d) $\lim_{x\to -2^-} f(x)$

e) $\lim_{x\to -2^+} f(x)$

f) $\lim_{x\to -2} f(x)$

g) $\lim_{x\to 1^-} f(x)$

h) $\lim_{x\to 1^+} f(x)$

i) $\lim_{x\to 1} f(x)$

j) $\lim_{x\to 4^-} f(x)$

k) $\lim_{x\to 4^+} f(x)$

l) $\lim_{x\to 4} f(x)$

Lesson 2: Evaluating Limits

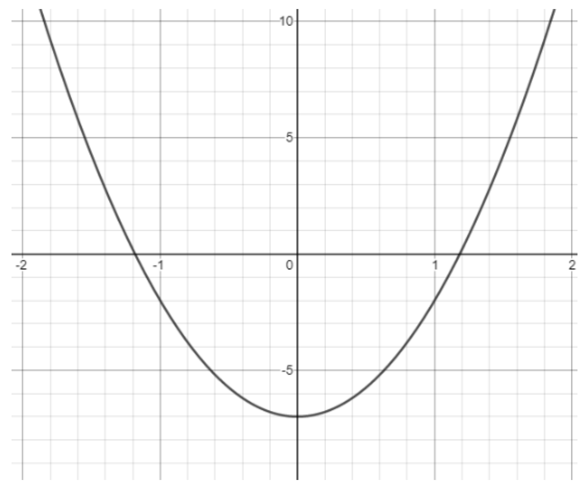
There are 5 ways to evaluate limits, each more treacherous than the last.

Note: If you can't solve a limit without the graph, you're a loser. You can use the graph to *help* you solve the limit, but don't rely on it.

Ways to Evaluate Limits

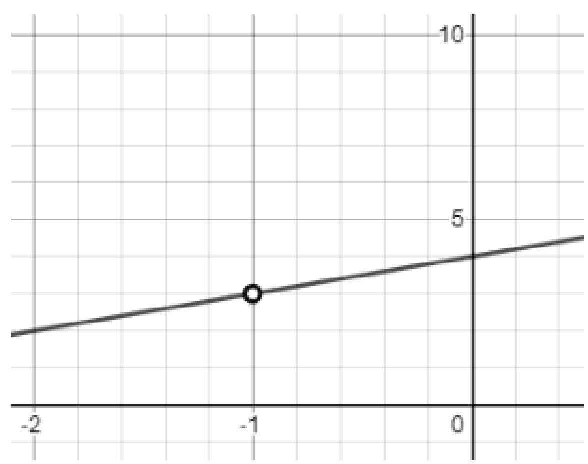
1) Direct Substitution:

$$\lim_{x\to 3}(5x^2 - 7)$$



2) Factor and Cancel:

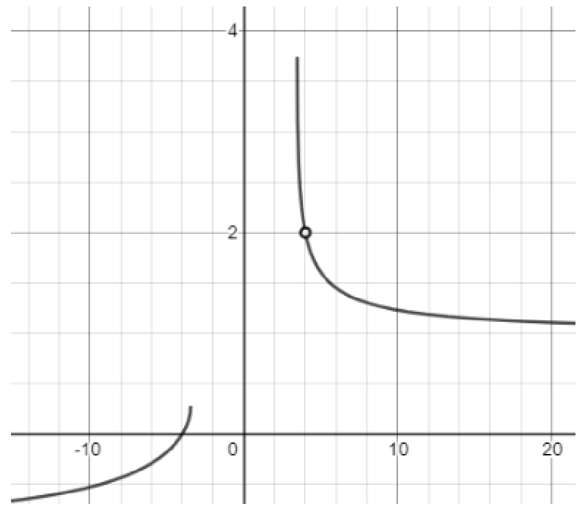
$$\lim_{x\to -1}\frac{x^2 + 5x + 4}{x + 1}$$



Ways to Evaluate Limits (cont.)

3) Rationalize and Cancel:

$$\lim_{t\to4}\frac{\sqrt{t^2-12}-2}{t-4}$$



4) Trig Limits:

$$\lim_{x\to0}\frac{\sin(x)}{x}=\lim_{x\to0}\frac{x}{\sin(x)}=1$$

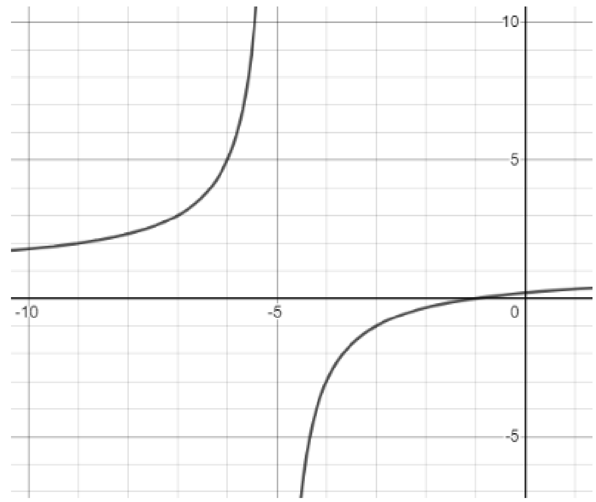
$$\lim_{x\to0}\frac{1-\cos(x)}{x}=\frac{\cos(x)-1}{x}=0$$

Memorize these!

Ways to Evaluate Limits (cont.)

5) Table of Values/Graph:

$$\lim_{x \to -5} \frac{x+1}{x+5}$$



x	y
-7	
-6	
-5	
-4	
-3	

Summary

When Evaluating limits, try the following strategies:

1) Direct Substitution
2) Factor and Cancel
3) Rationalize and Cancel
4) Trig Limits
5) Table of Values/Graph

Always try **Direct Substitution first**, and **Table of Values/Graph last**. The other methods depend on the type of problem, and will take a little practice to recognize which type they are.

Example Problems:

1. $\lim_{x \to 1}(5x^3 - 7x^2 + 10x + 4)$

2. $\lim_{x \to -4} \frac{x^2 - 16}{x + 4}$

3. $\lim_{x \to 0} \frac{\sin(2x)}{x}$

4. $\lim_{x \to 3} \frac{-2+\sqrt{x+1}}{x-3}$

5. $\lim_{x \to 1} \frac{1}{(x-1)^2}$

6. $\lim_{z \to 0} \frac{25-(z-5)^2}{z}$

7. $\lim_{h\to 0} g(h), \quad g(h) = \begin{cases} -h, & h < 0 \\ h^2, & h \geq 0 \end{cases}$

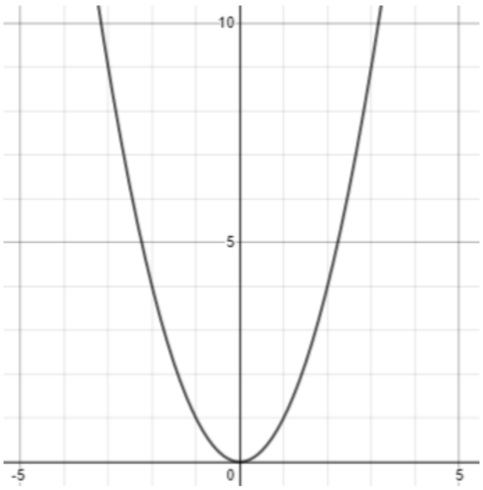
8. $\lim_{x\to 0} \frac{cotx\ sinx-1}{x}$

9. $\lim_{x\to 2} \sqrt{4-x^2}$

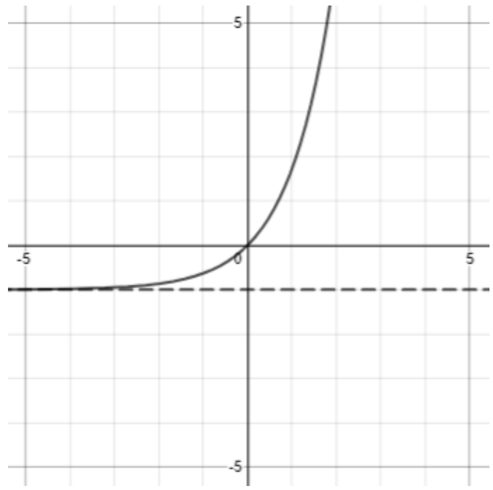
Lesson 3: Limits at Infinity

Limits at Infinity

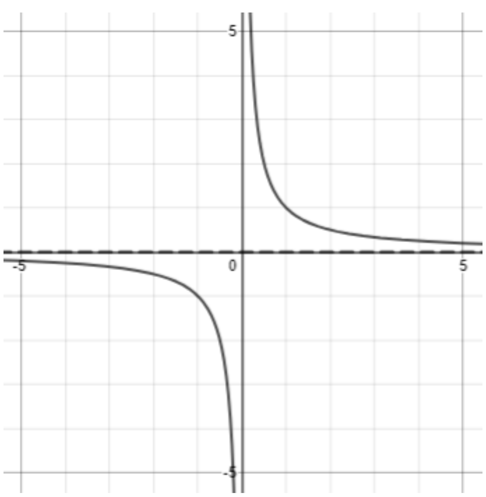
Limits at infinity describe the **end behavior** of a function (what happens at $x = \pm\infty$)



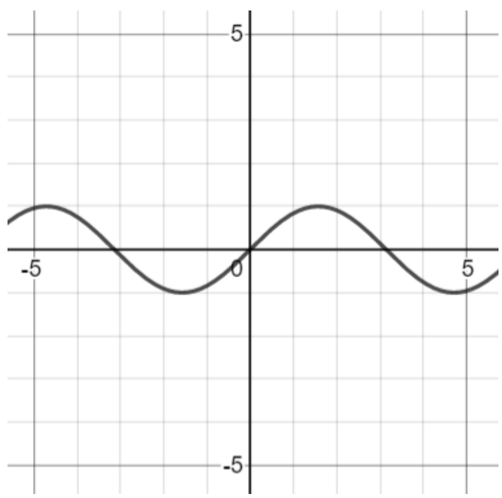
$f(x) = x^2$



$f(x) = e^x - 1$



$f(x) = \frac{1}{x}$



$f(x) = sinx$

Rules of Limits at Infinity

Let's look at some mathematical expressions involving infinity:

$\infty + 3 =$	$\frac{\infty^4}{\infty} =$
$\frac{1}{\infty} =$	$\frac{\infty^2 - 1}{3\infty^2 + \infty + 5} =$

Most limits at infinity problems involve a polynomial over another polynomial. If you forget what a polynomial looks like, it's probably because your math teachers and/or the public education system have failed you.

Here's what a polynomial over another polynomial looks like:

$$\frac{2x-5}{x^2+1} \qquad \frac{3x^2-5x}{4x^3+2x^2-x+7} \qquad \frac{-5x^4+x^2+3}{4x^6-12x^2}$$

There are 3 scenarios we can have with limits at infinity involving polynomials.
We will now cover each scenario.

$$\lim_{x\to\infty} \frac{x^2+2x+4}{x-5}$$

Degree of the numerator
is *greater than*
Degree of the denominator

$$\lim_{x\to\infty} \frac{2x^2-x+3}{x^3+4x+6}$$

Degree of the numerator
is *less than*
Degree of the denominator

$$\lim_{x\to\infty} \frac{x^2-9x-12}{3x^2+2x-3}$$

Degree of the numerator
is *equal to*
Degree of the denominator

For polynomials, $\lim_{x\to\infty} f(x)$ is always equal to $\lim_{x\to-\infty} f(x)$

Examples:

$$\lim_{x\to\infty} \frac{-x^2+6}{3x-4}$$

$$\lim_{x\to-\infty} \frac{-x^2+6}{3x-4}$$

This is **not** the case for piece-wise functions.

Example:

Find the limit as $f(x)$ approaches positive and negative infinity if

$$f(x) = \begin{cases} \dfrac{2x^2+1}{3x^2-6x+6}, & x > 2 \\ \dfrac{-4x}{x^2-5x+18}, & x < 2 \end{cases}$$

Limits at Infinity with Square Roots

First of all, these questions are the worst. They're difficult, very situational, and unlike most calculus problems, not fun at all.

Example:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{3x^2-2}$$

The worst-case scenario is if we have a square root, limit approaching negative infinity, and if the leading degree is odd.

Example:

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2 - 2x}}{x + 7}$$

Horizontal Asymptotes

To find the horizontal asymptotes of a function, you simply need to find the limits as x approaches ∞ and $-\infty$ (the two are usually the same answer).

Example Problems:

Find the horizontal asymptote (if any) of the functions:

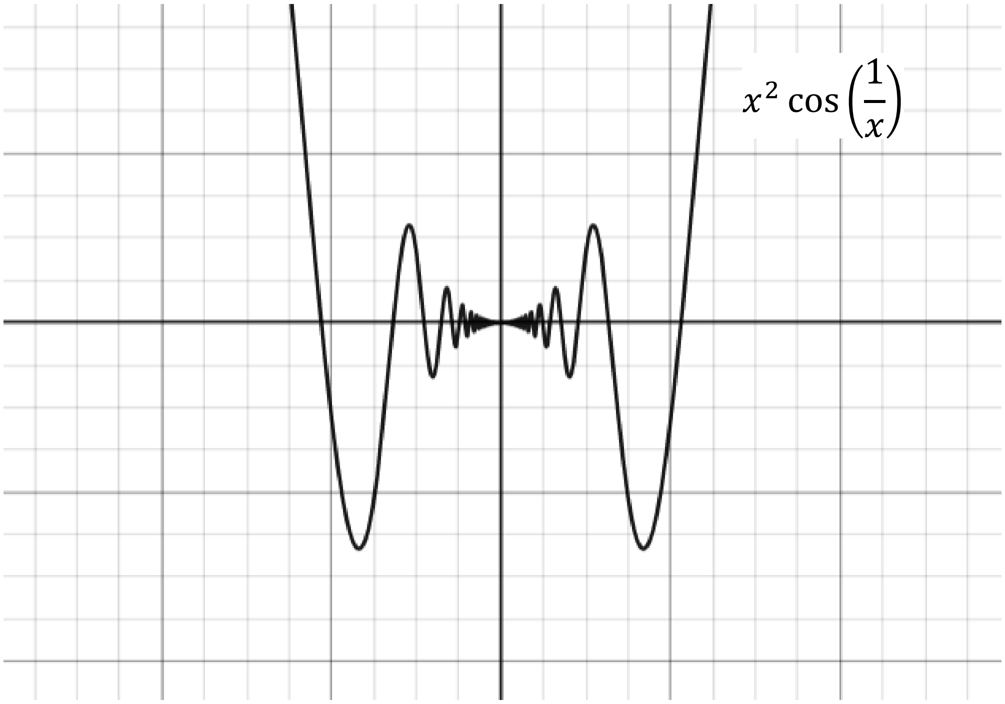
a) $y = \dfrac{3x^2 + 2x - 5}{-x^3 + 6x + 11}$

b) $f(x) = \dfrac{x^3 - 5x^2 + 12}{2x^2 + 5x^3}$

c) $g(x) = \begin{cases} \dfrac{4x + 2}{x^2 + 2x + 3}, & x > -3 \\ \dfrac{x + 1}{6x^2 + 2x + 5}, & x \leq -3 \end{cases}$

Sandwich/Squeeze Theorem

The Sandwich Theorem – or its less fun name, the Squeeze Theorem – is a method for solving limits.



This is what the Sandwich Theorem says:

$$h(x) \le f(x) \le g(x) \text{ AND } \lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L \text{ THEN } \lim_{x\to a} f(x) = L$$

In other words, if $f(x)$ is between two other functions $h(x)$ and $g(x)$ AND the limit for $h(x)$ and $g(x)$ equal the same number L, THEN the limit for $f(x)$ is also L.

For example, look at the function $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ as x approaches 0.

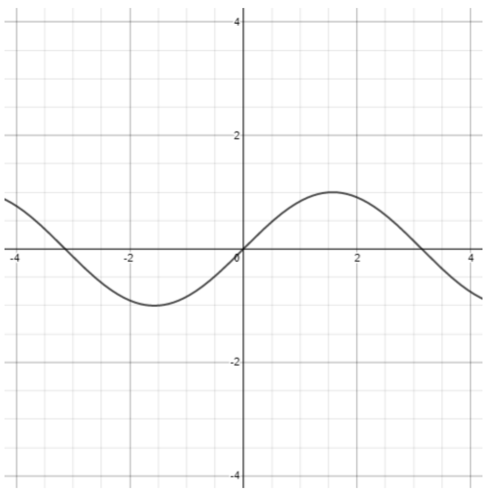
$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2 \qquad \lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

Therefore, $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

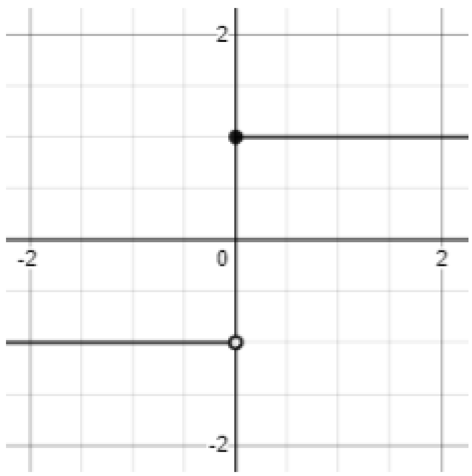
Lesson 4: Continuity

Continuity

Continuity is simple. Just ask yourself, “Can I draw the graph **without lifting the pencil**.”



$f(x) = \sin x$



$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Continuity Everywhere

Here are some examples of functions that are continuous for all values of x.

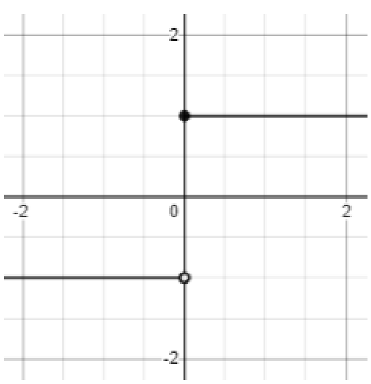
$f(x) = 5$ $\qquad$ $f(x) = \sin x$ $\qquad$ $f(x) = |x|$

Removable Discontinuities

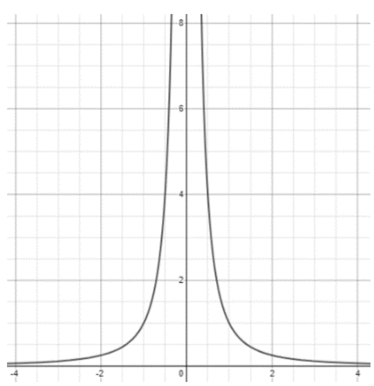
Removable discontinuities look like a hole in the graph. They are called *removable* discontinuities because they almost look continuous, except at one point. When you have a removable discontinuity, it means you're looking at a "Factor and Cancel" or a "Rationalize and Cancel" problem.

Non-Removable Discontinuities.

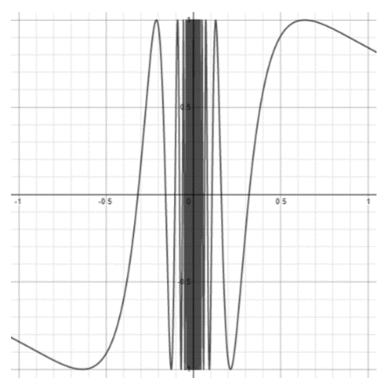
Jump Discontinuity



Infinite Discontinuity



Oscillating Discontinuity



Continuity at a point

We know $f(x)$ is continuous at $x = 2$ (or any point) if the following conditions are met:

a) $f(2)$ exists

b) $\lim_{x \to 2} f(x)$ exists

c) $\lim_{x \to 2} f(x) = f(2)$

Example Problems:

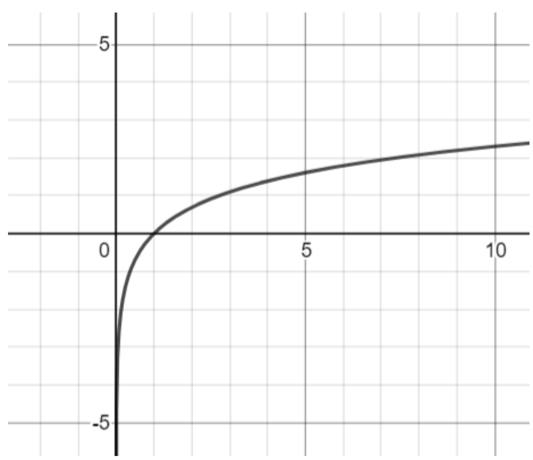
1. Is $f(x)$ continuous at $x = -1$?

$$f(x) = \begin{cases} x + 3, x \leq -1 \\ 2x^2, x > -1 \end{cases}$$

2. $f(x) = \begin{cases} 3x - 1\,, & x < 2 \\ 3\,, & x = 2 \\ -x + 7, & x > 2 \end{cases}$

Continuity over an interval. $f(x)$ is only continuous over some interval.

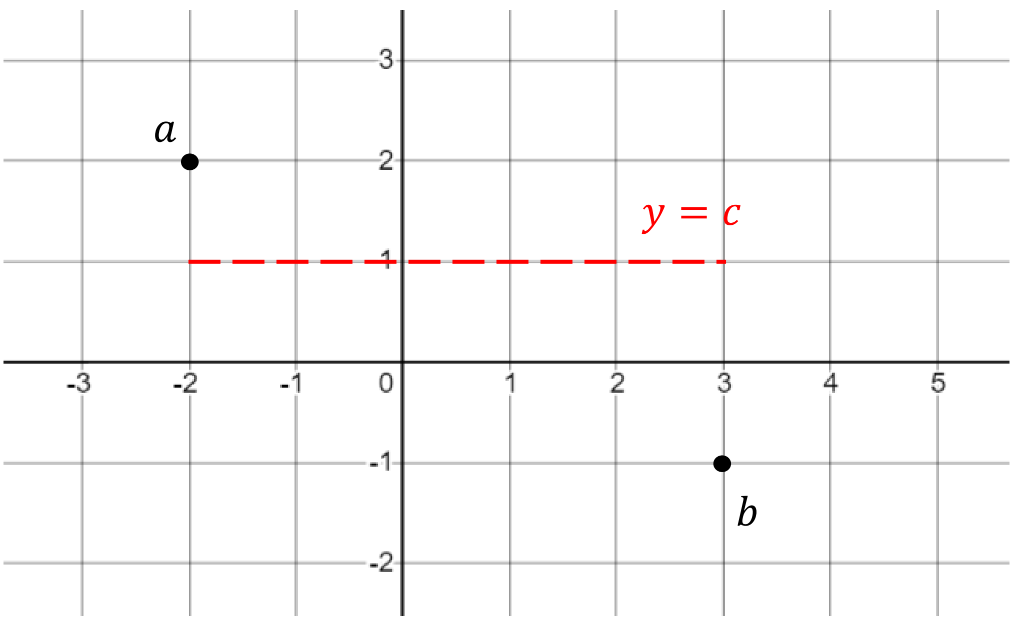
$f(x) = \ln x$



Lesson 5: Intermediate Value Theorem

Intermediate Value Theorem

The Intermediate Value Theorem is fairly useless and hard to explain in text, which is why I decided to explain it in song…



If you have a continuous function connecting points a and b, then there must be some point where the graph passes the line $y = c$ (where c is a y-value between a and b).

Example Problem:

Which of the following values of $f(x)$ **must exist** between $x = -2$ and $x = 3$ given that $f(-2) = 2$ and $f(3) = -1$ and $f(x)$ is continuous everywhere (circle all that apply):

a) -2

b) -1

c) 0

d) 1

e) 3

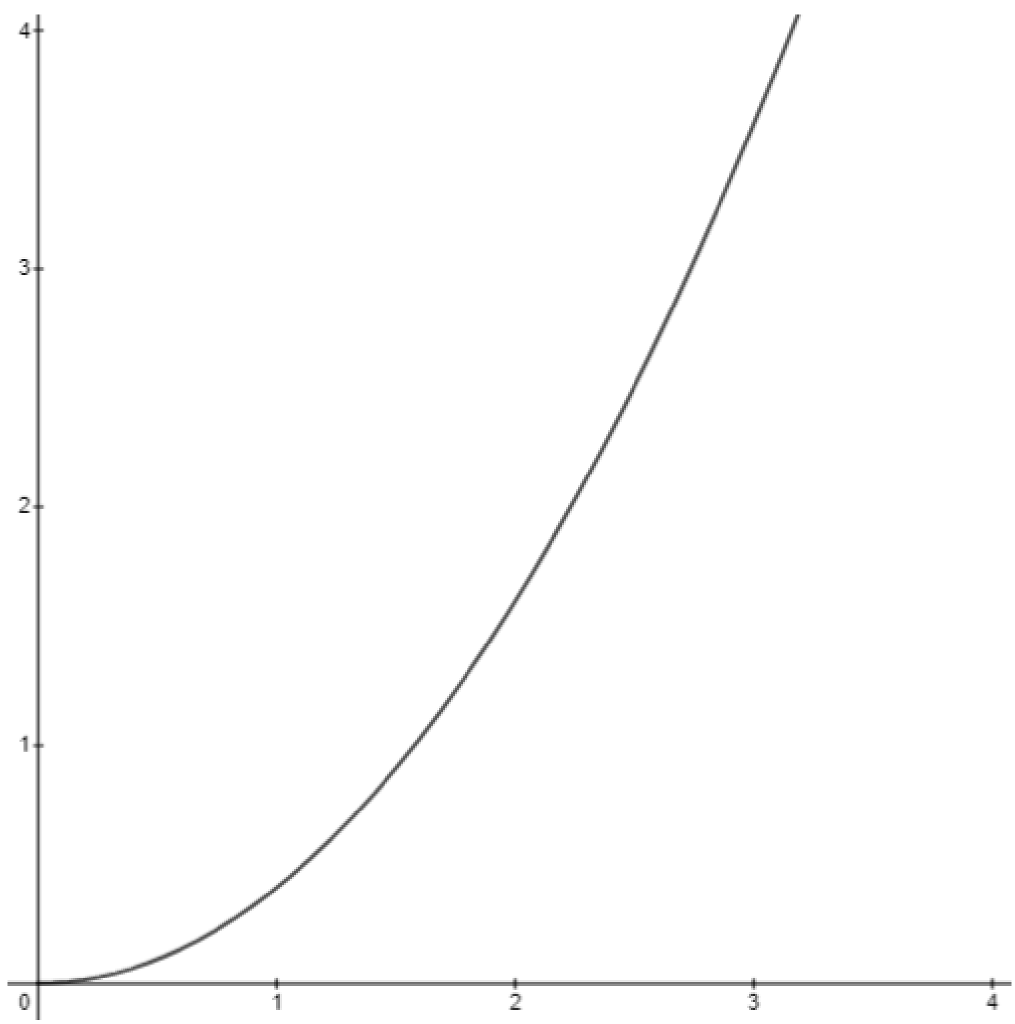
Note: The Intermediate Value Theorem is one of the **least** important topics in calculus. So don't stress out too much if you forget it.

Lesson 6: The Limit Definition of the Derivative

The Difference Quotient

The Difference Quotient is a formula which **computes the slope of a secant line** through two points: $(x, f(x))$ and $(x+h, f(x+h))$, where h is the distance along the x-axis from the first point to the second point:

$$Diff.Quotient = \frac{f(x+h) - f(x)}{h}$$



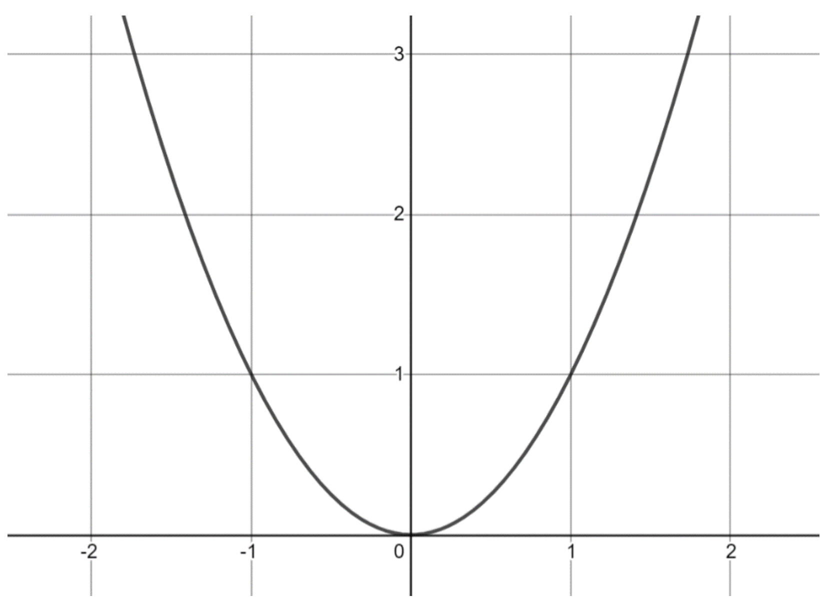
Example Problem:

Find the slope of the secant line between $x = 2$ and $x = 8$ of $f(x) = \sqrt{2x}$

Why is the Difference Quotient Important?

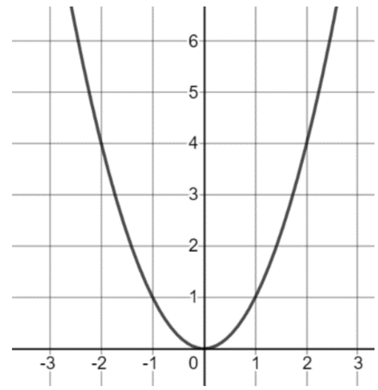
The Difference Quotient on its own is not important. However, it can be used to derive the **Limit Definition of the Derivative,** which is one of the most important discoveries in all of calculus.

Limit Definition of Derivative: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$



Example Problems:

1. What is the slope of the line $f(x) = x^2$ at x = 2?



2. What is the slope of the line $f(x) = \cos x$ at x = π?