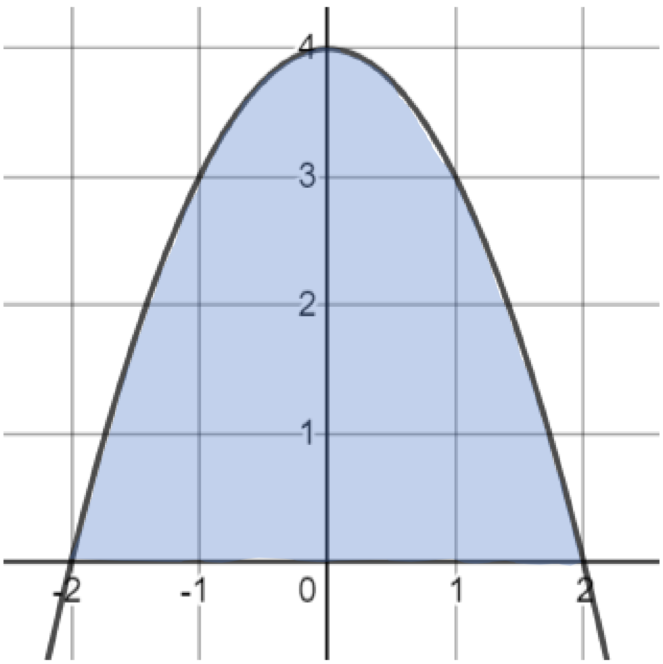
Lesson 27: Riemann Sums

Area Under a Curve

Let's say we wanted to find the area under the curve for this parabola:

$f(x) = -x^2 + 4$



We can get an estimate just by looking at the graph, but we don't have a formula for finding area under a parabola.

We can find the area of rectangles, triangles, trapezoids, and circles, but for complex shapes such as these, there is no formula.

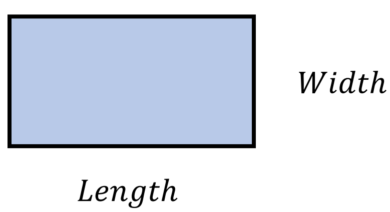
In this unit, we're going to discover how to find this area.

(Spoiler alert: the area under this parabola is $32/3$. I'll tell you how I got that soon enough.)

Rectangles

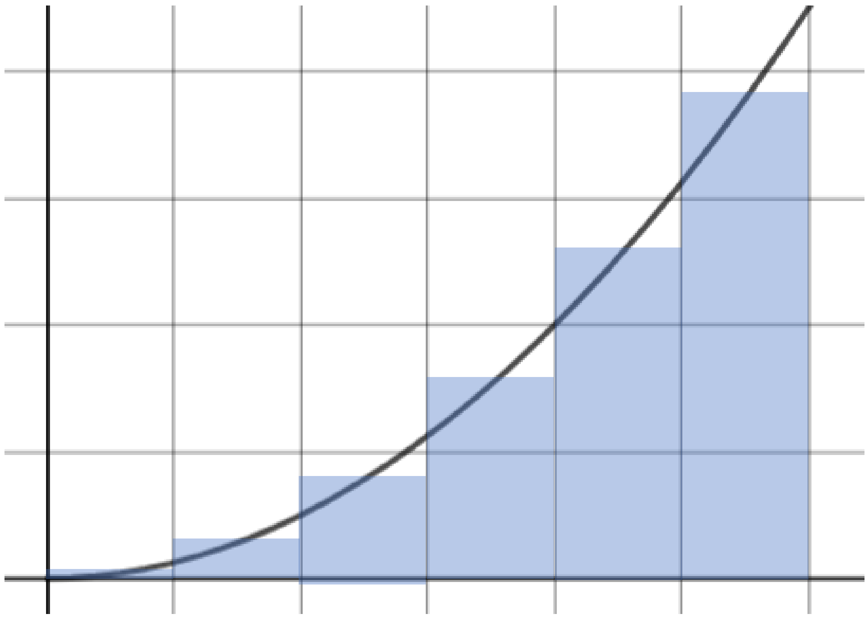
Hopefully, we all know the area of a rectangle by now.

$$Area = Length \times Width$$



I shouldn't even have to tell you that, but you'd be surprised. I once had a student taking college level calculus who didn't know the area of a rectangle. She ended up getting an A. To this day, I still don't understand how that was possible.

Now, imagine that instead of finding the area of a parabola, what if we found the area of a bunch of rectangles instead?



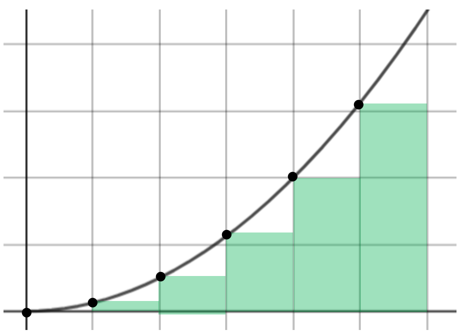
This is exactly what Alexander Riemann thought when he invented the Riemann sum.

(Don't look up the name Alexander Riemann. Just take my word for it.)

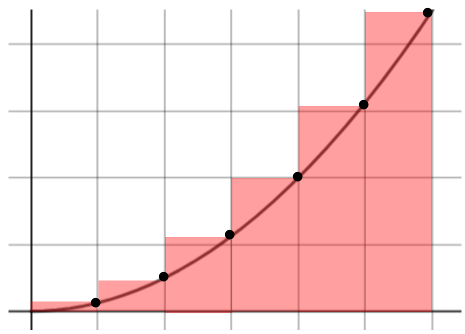
Riemann Sums

Riemann sums are great for approximating the area under a curve. It uses rectangles, which are easy to find the area, and we don't even need a graph. We can use Riemann sums with just an equation.

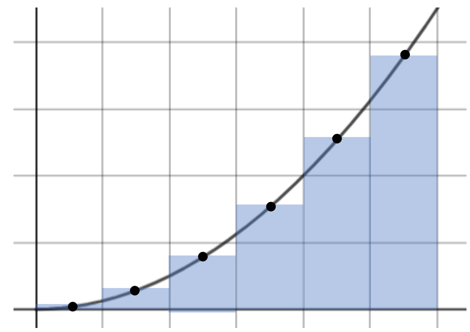
There are 3 kinds of Riemann sums, depending on where you anchor the height of the rectangle.



Left Riemann Sum



Right Riemann Sum



Midpoint Riemann Sum

Left Riemann Sums

$$Area \approx w[f(a)+f(a+w)+f(a+2w)+\cdots+f(a+(n-1)w)]\,,\;\; w=\frac{b-a}{n}$$

Right Riemann Sums

$$Area \approx w[f(a+w)+f(a+2w)+\cdots+f(a+(n-1)w)+f(a+nw)]\,,\;\; w=\frac{b-a}{n}$$

Midpoint Riemann Sums

$$Area \approx w\left[\frac{f(a)+f(a+w)}{2}+\frac{f(a+w)+f(a+2w)}{2}+\cdots+\frac{f(a+(n-1)w)+f(a+nw)}{2}\right]$$

What's b, a, and n in the equation?
b is the upper bound, *a* is the lower bound, *n* is the number of rectangles.

What does $f(a)$ and $f(a+w)$ and that stuff mean?
$f(a)$ is the y value at $x=a$. $f(a+3w)$ is the y value at the lower bound (a) plus 3 widths.

Can we see some examples?
Sure

Example Problems:

1. Find the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ from $x = 2$ to $x = 5$ with 3 rectangles of equal width.

2. Find the left, right, and midpoint Riemann sums for the function $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 3$ with 4 rectangles of equal width.

3. Use the table of values to find the left and right Riemann sums for $f(x)$ over $[0, 10]$. For each Riemann sum, use 4 rectangles of non-uniform length.

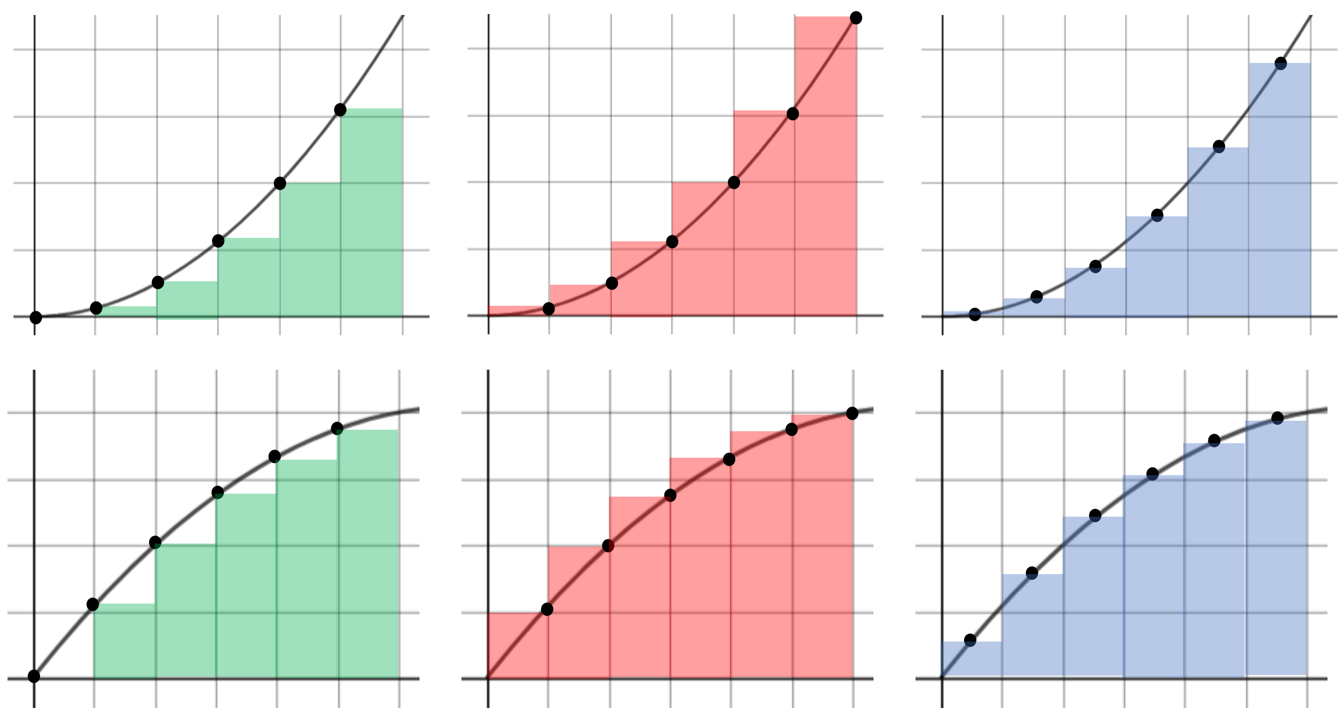
x	0	2	5	7	10
$f(x)$	2	6	12	9	3

Overestimates and Underestimates

As you may have already noticed, Riemann sums can be higher or lower than the actual area of the function.

Lucky for us, it's easy to tell whether or not a Riemann sum will be an overestimate or an underestimate.

For *Increasing* Functions



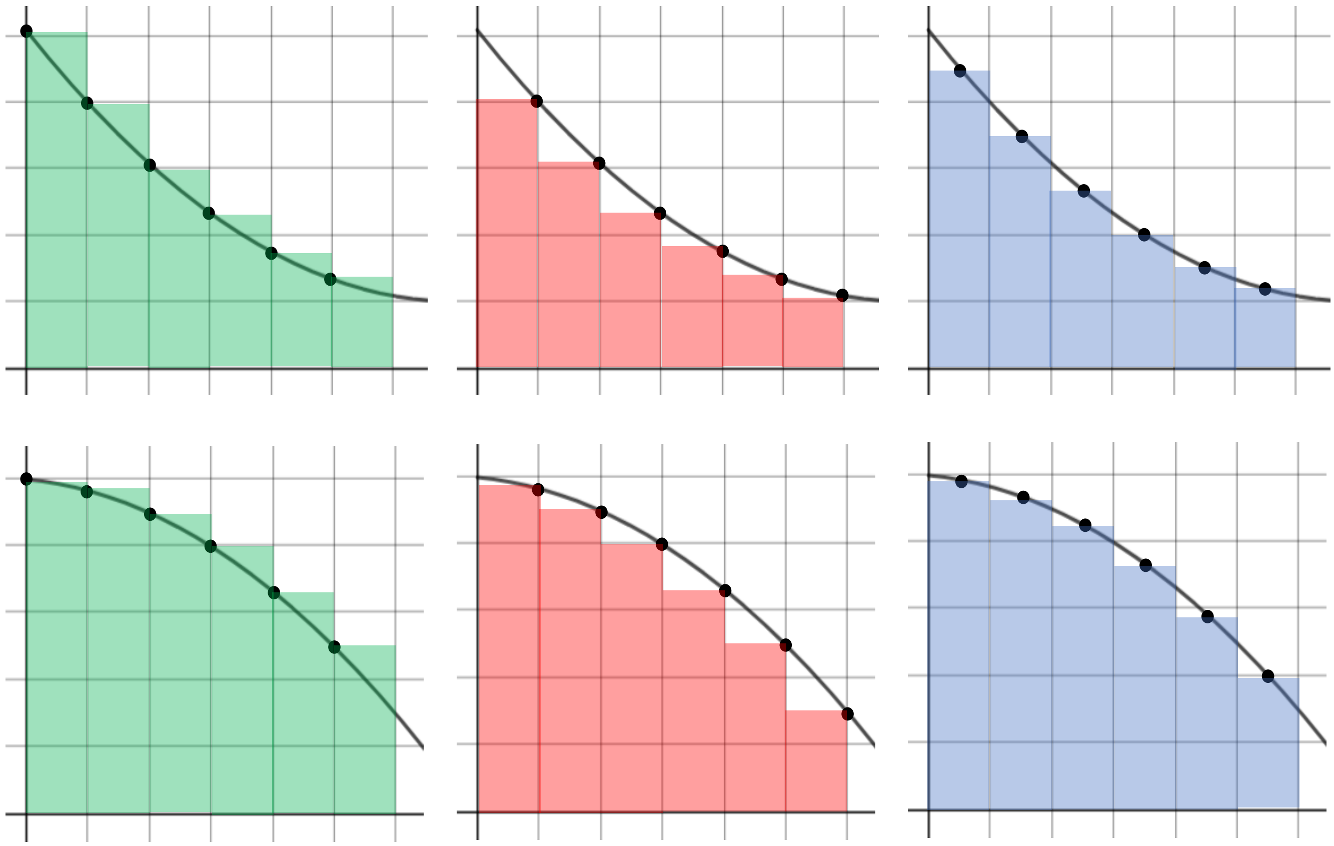
Left Riemann Sum is an Underestimate

Right Riemann Sum is an Overestimate

Midpoint Riemann Sum could be either

Note: We're never going to check if a **midpoint** Riemann sum is an overestimate or underestimate. We only care about **left** and **right** Riemann sums.

For *Decreasing* Functions



Left Riemann Sum is an Overestimate

Right Riemann Sum is an Underestimate

Midpoint Riemann Sum could be either

Example Problem:

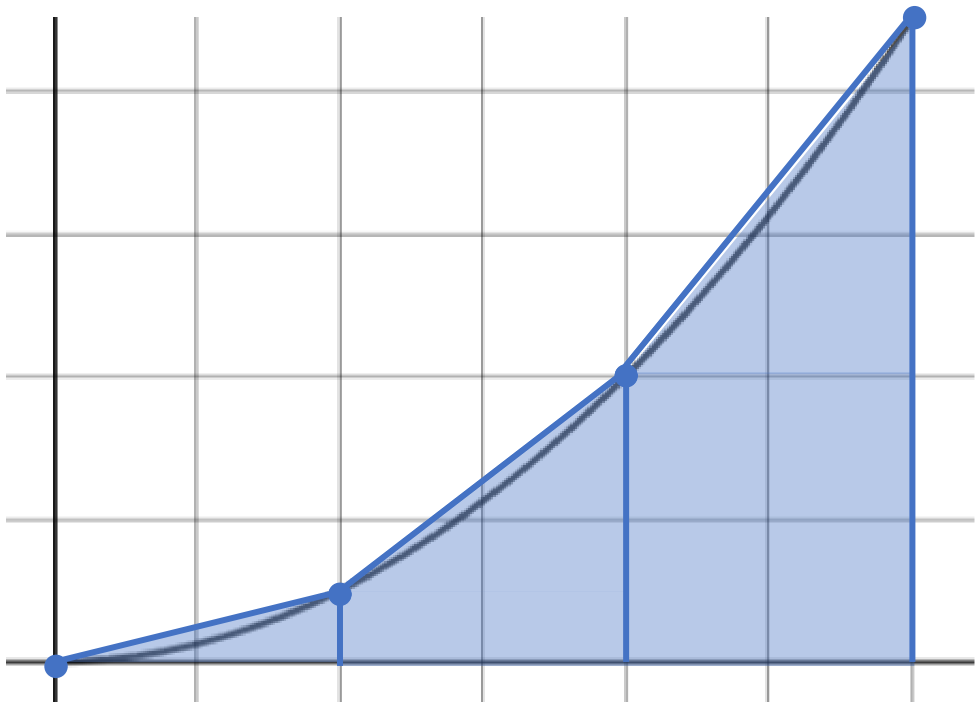
4. State whether the left and right Riemann sums for the function $f(x) = x^2$ from $x = 1$ to $x = 3$ are an overestimate or an underestimate.

Lesson 28: Trapezoidal Rule

Just like how Alexander Riemann discovered that we can use rectangles to approximate area, another man, Hans Trapezoid, discovered that we can get an even better approximation using trapezoids.

(Don't look up Alexander Riemann or Hans Trapezoid. Just take my word for it.)

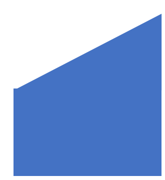
Example of Trapezoidal Rule with 3 Trapezoids



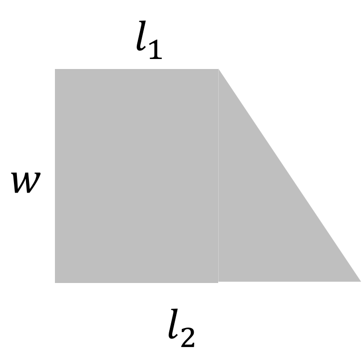
This is a trapezoid

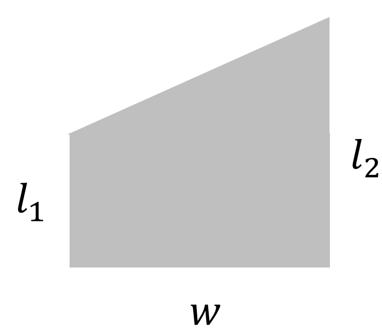
This is also a trapezoid

So is this



$$Area\ of\ a\ trapezoid = \left(\frac{l_1 + l_2}{2}\right)w$$





Trapezoidal Rule

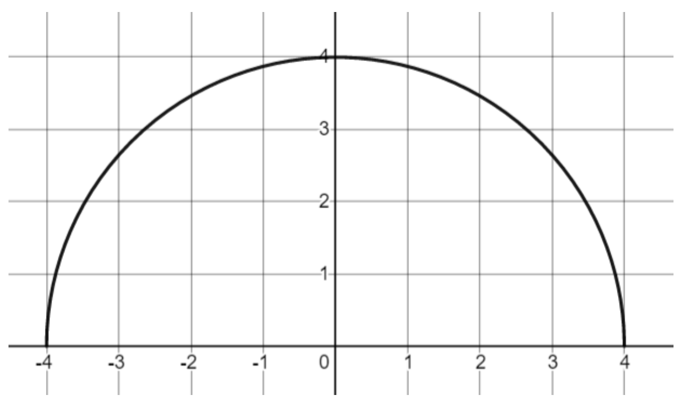
If you want to approximate the area of the function $f(x)$ on the interval $[a, b]$ with n trapezoids,

$$Area \approx \frac{1}{2}w[f(a) + \mathbf{2}f(a) + \mathbf{2}f(a+w) + \mathbf{2}f(a+2w) + \cdots + \mathbf{2}f(a+(n-1)w) + f(a+nw)]$$

$$w = \frac{b-a}{n}$$

Example Problems:

1. Approximate the area of the function $f(x) = \sqrt{16 - x^2}$ from $x = -4$ to $x = 4$ with 4 trapezoids of equal width.



2. Use the table of values to approximate the area of $f(x)$ over $[1, 11]$ with 4 trapezoids of non-uniform length.

x	1	4	5	8	11
$f(x)$	-1	3	2	6	7

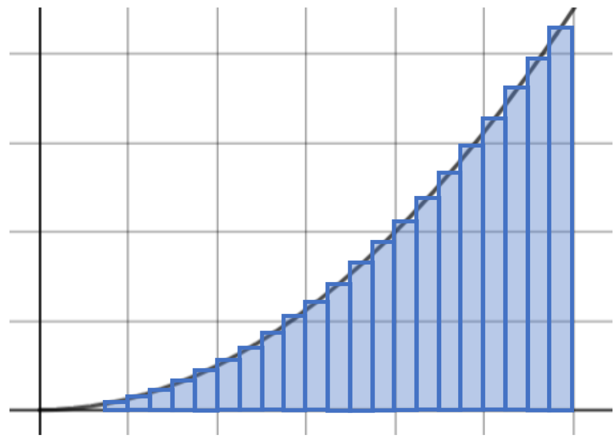
3. Given $f(2) = 4, f(3) = 12, f(4) = 9$, and $f(6) = 6$, approximate the area of $f(x)$ over $[2,6]$ using 3 trapezoids of non-uniform length.

Lesson 29: Properties of Integrals

The Integral

With Riemann sums, we were able to get an approximation of the area under the curve using rectangles.

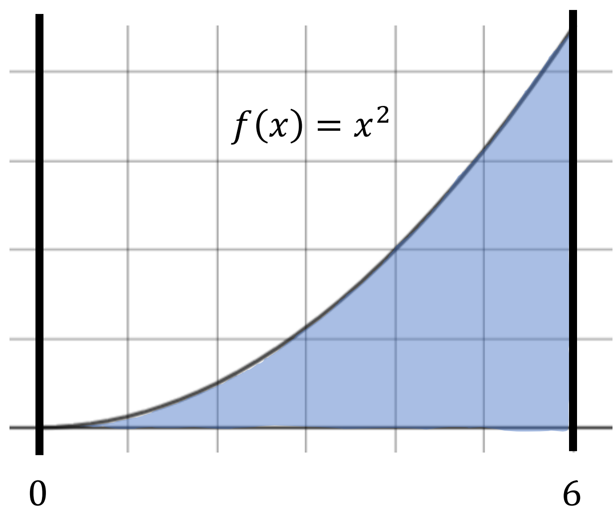
But what if we made a Riemann sum with many rectangles each with a very small width?



And what if we had infinitely many rectangles each with an infinitesimally small width? This is essentially the definition of an *integral.*

Integrals are how we find areas for shapes of functions. Just like derivatives, integrals have their own set of rules to solve for them.

Conveniently enough, the rules of integration are the mirror opposite of the rules for derivatives. This is why the integral is also called the *anti-derivative.*



How would we write the integral for the graph to the right?

Answer:

$$\int_0^6 x^2\,dx$$

Upper bound

Long S means "integral"

$$\int_0^6 x^2\,dx$$

Function

Lower bound

Oh jeez, so this explanation will be long. Think of it this way: the function is the height of the rectangle and the "dx" is the width. ***Don't worry about what dx means.*** *Just know that the x should match the variable in the function and that every integral needs to have d-something (such as dx, dy, or du).*

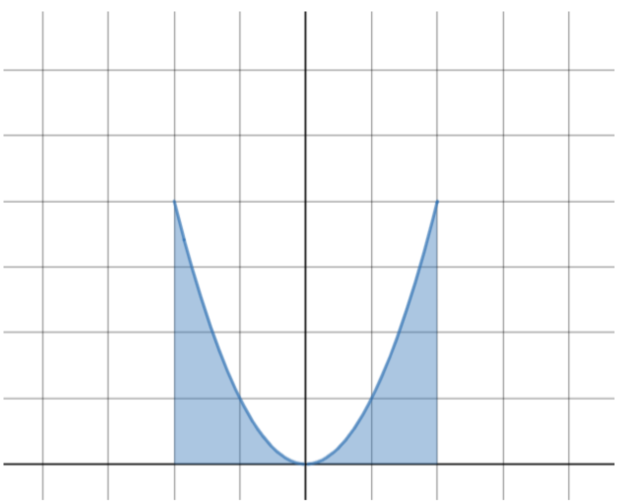
Definite vs. Indefinite Integrals

The integral we just saw was a *definite integral*. The other type of integral is an *indefinite integral*.

What's the difference between a definite and an indefinite integral?

Definite integrals have bounds

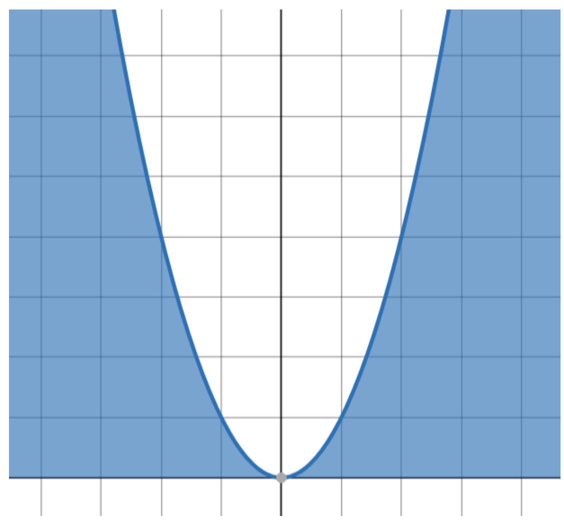
$$\int_{-2}^{2} x^2 \, dx$$



Only counts the area in the bounds

Indefinite integrals do not

$$\int x^2 \, dx$$



Counts all area between $-\infty$ and $+\infty$

Properties of Integrals

These are properties

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b 5f(x)dx = 5\int_a^b f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

These are misconceptions

$$\int_a^b f(x)\cdot g(x)\,dx \neq \int_a^b f(x)dx \cdot \int_a^b g(x)\,dx$$

$$\int_a^b (f(x)+1)\,dx \neq \int_a^b f(x)dx + 1$$

$$\int_a^b (f(x))^2 dx \neq \left(\int_a^b f(x)dx\right)^2$$

Example Problems:

1. Let $\int_1^4 f(x)dx = 3$, $\int_4^7 f(x)dx = 12$, and $\int_1^7 g(x)dx = -5$

a) $\int_4^1 f(x)dx$

b) $\int_1^7 f(x)dx$

c) $\int_4^7 3f(x)dx$

d) $\int_1^7 \big(f(x)+g(x)\big)dx$

e) $\int_{1}^{7} \left(2f(x) - 6g(x)\right)dx$

f) $\int_{1}^{1} f^2(x)\, dx$

2. Let $\int_{-1}^{5} h(x)dx = 6$, $\int_{0}^{5} h(x)dx = 7$, and $\int_{-1}^{5} k(x)dx = \sqrt{3}$

a) $\int_{-1}^{0} h(x)dx$

b) $\int_{0}^{5} h(x)dx$

c) $\int_{-1}^{5} \frac{k(x)}{2} dx$

d) $\int_{5}^{-1} -h(x)dx \cdot \int_{-1}^{5} \sqrt{3}k(x)dx$

Lesson 30: Basic Rules of Integration

Indefinite Integrals

Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^4 dx$$

Example Problems:

1. $\int x \, dx$

2. $\int x^2 \, dx$

3. $\int x^5 \, dx$

4. $\int 16 \, dx$

5. $\int (2x + 3) \, dx$

6. $\int \sqrt{x} \, dx$

7. $\int \frac{1}{x^2} \, dx$

8. $\int x^{4/3} \, dx$

9. $\int \frac{2x-4}{5}\, dx$

10. $\int \left(\frac{3}{2}x - \frac{4x^{7/6}}{3} + \frac{6}{x^{1/3}}\right) dx$

Trig Integrals

$$\int \cos x\, dx = \sin x + C$$

$$\int \sec x \tan x\, dx = \sec x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \csc x \cot x\, dx = -\csc x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \csc^2 x\, dx = -\cot x + C$$

It's easy to get the trig **derivatives** mixed up with the trig **integrals**. Make sure you dedicate the time to remembering them, unless you want to fail the test.

(You'd be surprised. Some kids want to fail the test. That's why they don't study.)

Lesson 31: Definite Integrals

Now, we're going to talk about definite integrals. These are integrals **with bounds.**

There's one big difference between definite and indefinite integrals, and that's the *+C* part.

Definite Integrals

$$\int_0^5 x\,dx$$

$$= \left(\frac{1}{2}x^2\right)\Bigg|_0^5$$

$$= \frac{1}{2}(5)^2 - \frac{1}{2}(0)^2$$

$$= \frac{25}{2} - 0$$

$$= \frac{25}{2}$$

We solve Definite Integrals just like Indefinite Integrals. The only difference is that we plug in the top value for x and subtract that from the bottom value for x.

The integral rules are exactly the same. The only difference is that *+C* gets replaced with a long bar and the bounds.

Example Problems:

1. $\int_{-1}^{4} 3x^2 dx$

2. $\int_{-1}^{1} \sqrt[3]{x}\, dx$

3. $\int_{-4}^{0} |2x+1| dx$

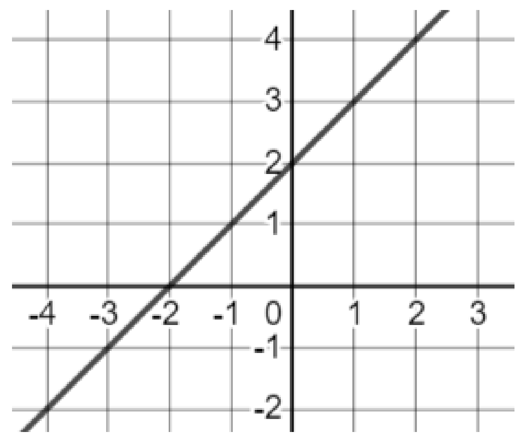
Finding Areas of Shapes using Integrals

Sometimes, we can solve integrals using the area formula of simple shapes like rectangles, triangles, and semi-circles.

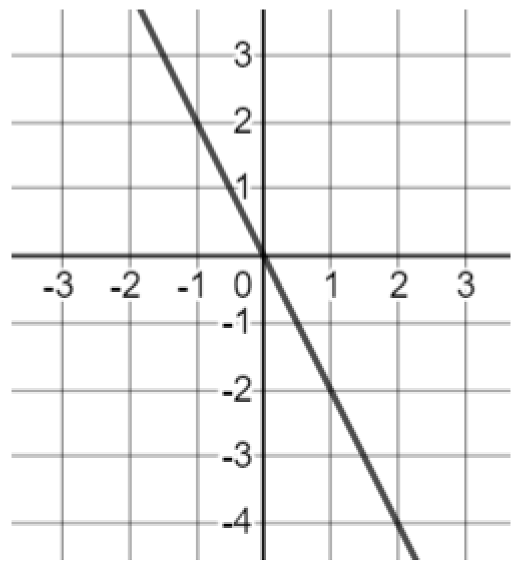
In order to do this, the function has to resemble a geometric shape (in other words, you have to graph it and know the area of that shape).

Example Problems:

1. $\int_{-2}^{2}(x+2)dx$



2. $\int_{-1}^{2}-2x\,dx$



3. $\int_0^5 5\,dx$

4. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

Lesson 32: U-Substitution

U-Substitution is one of the most useful and difficult methods of solving integrals. It will take a while for you to get the hang of it, so just keep trying!

What is the U-Substitution?

Here's a quick example of an integral solved using U-Substitution:

$$\int (2x+1)^3\,dx$$

When to use the U-Substitution

If you haven't noticed by now, we haven't covered any integral problems with **product rule, quotient rule, or chain rule.** That's because those rules don't exist with integrals.

Instead, whenever it looks like there's a product rule, quotient rule, or chain rule, we're going to use a U-Substitution instead.

How to use U-Substitution

1. Pick an expression to replace with u
2. Find du/dx (the derivative of u) and solve for dx
3. Substitute u and dx into the integral
4. Simplify and cancel out any x terms (if there are any x terms left, it usually means you need to pick another expression for u)
5. Take the integral with u instead of x
6. After you take the integral, substitute x back in for u

Note: I usually use the "Goldilocks Principle" when selecting my expression for u (not too long, not too short, just right).

Example Problems:

1. $\int (-4x+6)^5 \, dx$

2. $\int_0^1 2x\,(x^2+2)^4 \, dx$

3. $\int \frac{x^2 + x}{(4x^3 + 6x^2)^3}\, dx$

4. $\int \tan^4 x \sec^2 x\, dx$

5. $\int_0^3 x(x+1)^5 \, dx$

Lesson 33: Exponential and Logarithmic Integrals

Integrals of Exponential Functions

$$\int e^x \, dx = e^x + C$$

Note: If the exponent has more than just x (such as $e^{2x}, e^{3x-1}, e^{x^2+x+5}$) then you should use a U-Substitution to get it in the form

$$\int e^u \, du = e^u + C$$

Example Problems:

1. $\int e^{2x} \, dx$

2. $\int e^{-5x+13} \, dx$

3. $\int 2x^2 \, e^{x^3+6} \, dx$

4. $\int \frac{e^{3x}}{e^{8x+1}} \, dx$

If you've been paying attention, there's one very simple integral we haven't seen yet…

$$\int \frac{1}{x}\,dx$$

Integrals into Logarithmic Functions

$$\int \frac{1}{x}\,dx =$$

$$\int \frac{1}{u}\,du =$$

Why absolute value signs?

Example Problems:

5. $\int \frac{\sqrt{3}}{4x}\,dx$

6. $\int \tan x\ dx$

Lesson 34: Inverse Trig Integrals

Inverse Trig Integrals

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \quad \Big| \quad \int \frac{1}{1+x^2}dx = \tan^{-1} x + C \quad \Big| \quad \int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C$$

Note: You never have to use trig integrals with cos^{-1}, cot^{-1}, or csc^{-1}.

But what if you don't have exactly what's above?

$$\int \frac{1}{\sqrt{8-2x^2}}dx$$

Example Problems:

1. $\int \frac{3}{\sqrt{1-4x^2}}dx$

2. $\int \frac{dx}{9x^2+9}$

3. $\int \frac{1}{2x\sqrt{4x^2 - 25}}\,dx$

Lesson 35: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) has 2 parts. The first part is intuitive and shouldn't come as a surprise, and the second part we already covered.

Question: if it's called the *Fundamental* Theorem of Calculus, why didn't we learn it sooner?

That's a good question…

FTC Part 1

The derivative of an integral is just the original function

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x) \qquad \frac{d}{dx}\int_c^x \sqrt{t+1}\,dt = \sqrt{x+1}$$

Note: We use $f(t)$ inside the integral instead of $f(x)$ because the variable inside the integral cannot use the same variable as the bounds.

If there's more than just x in the bounds, use this form instead (because we need to account for the chain rule)

$$\frac{d}{du}\int_c^u f(t)\,dt = f(u)\cdot u' \qquad \frac{d}{dx}\int_c^{x^2} \sqrt{t+1}\,dt = \sqrt{x^2+1}\cdot 2x$$

FTC Part 2

Definition of the definite integral

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Note: $F(x)$ is the integral of function $f(x)$. In other words, $F'(x) = f(x)$.

If you want to see example problems on FTC Part 2, see the lesson on Definite Integrals (Lesson 31).

This lesson will just be focusing on FTC Part 1.

Example Problems:

1. $\frac{d}{dx}\int_0^x t\,dt$

2. $\frac{d}{dx}\int_{-2}^x \frac{dt}{t}$

3. $\frac{d}{dx}\int_x^{\pi} \sin t\,dt$

4. $\frac{d}{dx}\int_2^{x^3} \ln t\,dt$

5. $\frac{d}{du}\int_{-1}^{u^2+u} \frac{t^3-1}{t}\,dt$

6. $\frac{d}{dt}\int_{-t+2}^{t^4} w\,dw$