Problem Set 6: The Limit Definition of the Derivative

Find the derivative of the function using the Limit Definition of the Derivative.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

1. $f(x) = x + 3$

$f(x+h) = (x+h)+3$

$$f'(x) = \lim_{h\to 0} \frac{x+h+3-(x+3)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h+\cancel{3}-\cancel{x}-\cancel{3}}{h} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} = 1$$

2. $f(x) = x^2$

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} (2x+h) = 2x$$

3. $f(x) = 2x^2 - 1$

$f(x+h) = 2(x+h)^2 - 1 = 2(x^2+2xh+h^2)-1 = 2x^2+4xh+2h^2-1$

$$f'(x) = \lim_{h\to 0} \frac{2\cancel{x^2}+4xh+2h^2-\cancel{1}-(2\cancel{x^2}-\cancel{1})}{h}$$

$$= \lim_{h\to 0} \frac{4xh+2h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4x+2h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (4x+2h)$$

$$= 4x$$

Find the slope of the function at the indicated point using the Limit Definition of the Derivative.

4. $f(x) = \sqrt{x}$ at $x = 4$

$$f(x+h) = \sqrt{x+h}$$

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

5. $f(x) = \sin x$ at $x = \pi$

$$f(x+h) = \sin(x+h)$$

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\left(\cos(h) - 1\right) + \sin(h)\cos(x)}{h}$$

$$= \sin(x)\underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{=0} + \cos x \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{=1}$$

$$= 0 \cdot \sin x + 1 \cdot \cos x$$

$$f'(x) = \cos x$$

$$f'(\pi) = \cos(\pi)$$

$$= -1$$